

# Exam 1

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*Note:* This is an edited version of the exam that was given to the students.

**Read these instructions carefully.**

You have one hour, fifty minutes for the exam. Each of the questions are worth 10 points total.

You have your choice of doing either problem 9 or problem 10. You must do one of those two problems. In the table below, cross out the problem that you do **not** want graded.

Additionally, you have your choice of doing either problem 11 or problem 12. You must do one of those two problems. In the table below, cross out the problem that you do **not** want graded.

<b>Problem</b>	<b>Score</b>
1	
2	
3	
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8	
9	
10	
11	
12	
<b>Total:</b>	

## 1 Quick shot, part 1.

How many ways can you roll three distinguishable dice so the minimum number of the three is 1? Put differently, how many ways can you roll three distinguishable dice so at least one of them shows a 1?

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We condition on having exactly one 1, exactly two ones, or exactly three ones. With one 1, there are  $\binom{3}{1}(5)^2 = 75$  different ways, for we first choose which die has the 1, then we have 5 different choices for each of the other two die, as neither of them can be one. Similarly, we have  $\binom{3}{2}(5) = 15$  ways to have exactly two ones, and  $\binom{3}{3} = 1$  way to have exactly three ones. Hence, there are 91 ways total.

## 2 Quick shot, part 2.

How many subsets of  $[10]$  contain precisely two odd numbers?

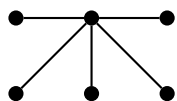
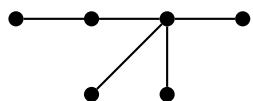
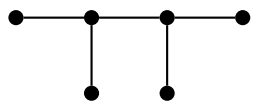
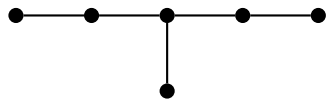
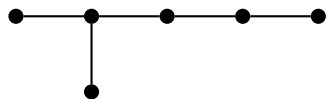
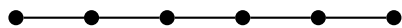
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There are  $\binom{5}{2}$  ways to choose the two odd numbers. Additionally, we may have any of the 5 even numbers that are available, of which there are  $2^5$  different subsets. Hence, there are  $\binom{5}{2}2^5 = 320$  different subsets of  $[10]$  containing precisely two odd numbers.

### 3 Draw some pictures.

Find (and draw) all non-isomorphic unlabeled trees on 6 vertices.

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## 4 Counting matrices.

(a) How many  $m \times m$  square matrices are there where each entry in the matrix is either 0 or 1?

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There are  $m^2$  entries in the matrix, each of which is 0 or 1. Hence, there are  $2^{m^2}$  different matrices.

(b) How many  $m \times m$  0/1 square matrices are there with the added condition that exactly one row is either all zero or all one?

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We first choose the exact row that we want to be all zero or all one. There are  $m$  different ways to do this. We have two choices for what this row could be - all zero or all one. For the  $m - 1$  other rows, each whole row has  $2^m - 2$  different possibilities, since it can be whatever it wants to be, but it can't be all zero or all one. Putting it all together, we have

$$2m(2^m - 2)^{m-1}$$

different possibilities.

(c) What is the probability that a  $m \times m$  0/1 square matrix has exactly one row that is all zero or all one? What happens to this probability as  $m \rightarrow \infty$ ?

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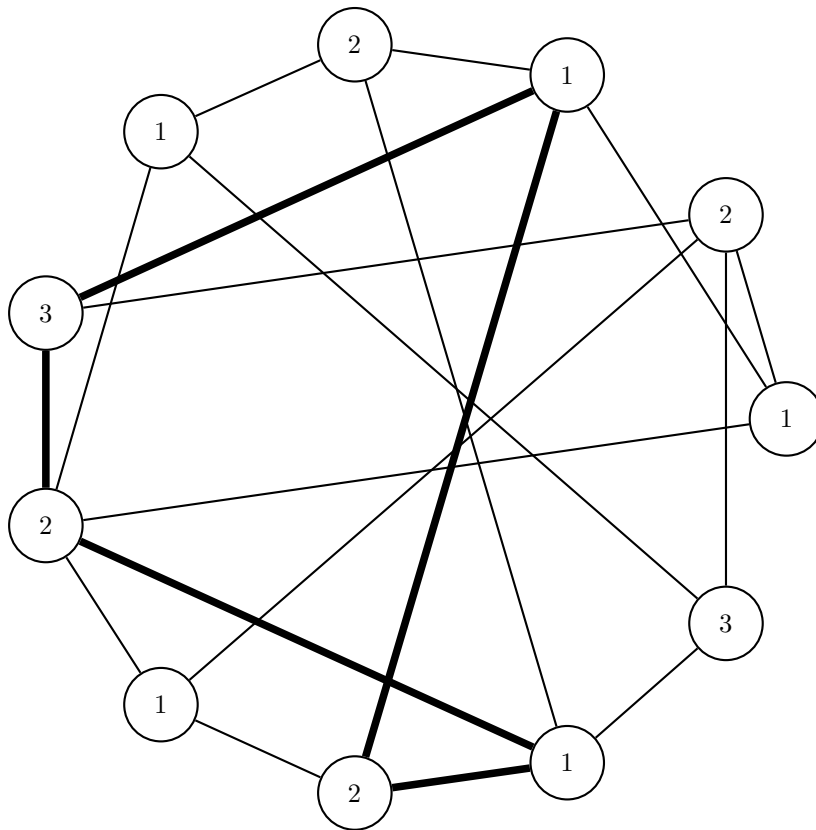
We need to compute:

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{2m(2^m - 2)^{m-1}}{2^{m^2}} &\leq \lim_{m \rightarrow \infty} \frac{2m2^{m(m-1)}}{2^{m^2}} \\ &= \lim_{m \rightarrow \infty} \frac{2m}{2^m} \\ &= 0 \end{aligned}$$

The last equality showing that it is 0 can be justified by l'Hôpital's Rule, for example. Since the limit is less than or equal to zero, and it is non negative, it must be equal to zero.

## 5 Chromatic number.

Find the chromatic number of the following graph. Explain why the chromatic number can't be any lower.  
*Note:* An extra copy of this graph is given at the end of the exam. Abuse that one and leave this copy clean.



You can color this graph with three colors - an example is shown above. You can't color it with less than three colors since there is an odd cycle, as shown above. A graph is bipartite if and only if it has no odd cycles, so we can't be bipartite!

## 6 Combinatorial proof.

If  $n$  is even, give a combinatorial proof that the number of  $n \times n$  0/1 matrices is the same as the number of  $\frac{n}{2} \times \frac{n}{2}$  matrices so that each entry is a number between 0 and 15, inclusive.

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Consider a  $\frac{n}{2} \times \frac{n}{2}$  matrix. For each number in the matrix, we associate it with a 4-length bit string, arranged in a 2-by-2 matrix. So, for example,

$$\begin{aligned} 0 &\rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 1 &\rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ 2 &\rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &\vdots \\ 15 &\rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

So replace each number in the  $\frac{n}{2} \times \frac{n}{2}$  matrix with its associated 2-by-2 matrix, and what is left is a  $n \times n$  0/1 matrix. This process is reversible, so we have the same amount in each one.

## 7 *In-n-Out.*

(a) *In-n-Out* has three different types of burgers on their menu: the hamburger, the cheeseburger, and the “double-double”, which is a double-cheeseburger. Duke, the patriarch of a voracious family, is sent to *In-n-Out* to order 15 burgers total, with the stipulation that there are at least two burgers of each kind. How many different ways can Duke order?

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Duke only has free will over 9 of the burgers, since we have to set aside 6 burgers to make sure we have 2 of each of the 3 different kinds. So this problem is equivalent to the number of ways to get 9 burgers with no restrictions. There are three different types of burgers to get, so we have

$$\binom{9 + 3 - 1}{9} = 55 \text{ ways.}$$

(b) *In-n-Out* then introduces the ability to make each different kind of burger “animal style,” which involves adding pickles, grilled onions, and special sauce. Once again Duke is sent to *In-n-Out* to order 15 burgers with the same stipulation that he must return with at least 2 of each kind. How many different ways can Duke order?

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As before, we only have free will over 3 of the hamburgers, since now we have six different types and we have to get 2 of each. So we have 3 burgers and 6 different types, so we have

$$\binom{3 + 6 - 1}{3} = 56 \text{ ways.}$$

## 8 More binomial identities.

Show that, for any  $n \geq 0$ ,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

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Using the binomial theorem, we see

$$\begin{aligned} \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} &= \sum_{k=0}^n \binom{n}{k} (-1)^k \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^k (1)^{n-k} \\ &= (1 - 1)^n \\ &= 0 \end{aligned}$$

## 9 Just an example.

A consequence of the Pigeonhole Principle is the following theorem:

**Theorem (Erdős-Szekeres 1935)**

Given a sequence of  $n^2 + 1$  distinct integers, either there is an increasing subsequence of  $n + 1$  terms or a decreasing subsequence of  $n + 1$  terms.

Show that this theorem is the best possible in the following way: show that for every  $n$ , there is a sequence of  $n^2$  integers with *no* increasing subsequence of  $n + 1$  terms AND *no* decreasing subsequence of  $n + 1$  terms.

*Hint:* (This is a general hint for all problems.) Try it for small  $n$ . Find a pattern.

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For  $n = 2$  we have 2143.

For  $n = 3$  we have 321654987.

For general  $n$ , we need a sequence of length  $n^2$ . We can write the first  $n$  numbers in reverse, then we can write the numbers  $n + 1, \dots, 2n$  in reverse, and so on. It would look as follows:

$$n(n-1) \dots 21(2n)(2n-1) \dots (n+1)(3n)(3n-1) \dots (2n+1) \dots (n^2)(n^2-1) \dots ((n-1)n+1).$$

## 10 Just another example.

Let  $G$  be a graph with  $n$  vertices. In class we mentioned that the chromatic polynomial  $P(G, x)$  must satisfy the following, among other things:

1. The degree of  $P(G, x)$  is  $n$  and the leading coefficient (the coefficient of  $x^n$ ) is 1.
2. The coefficient of  $x^{n-1}$  is  $-|E(G)|$ .
3. The coefficient of the constant term is zero.

Show that this isn't enough - give an example of a polynomial satisfying the three conditions above that is *not* the chromatic polynomial of any graph. Justify your answer.

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One such example is  $x^3 - 2x^2$ . This would imply that our graph had 3 vertices and 2 edges, and there is only one such graph: the path. However, we know that this graph has chromatic polynomial of  $x^3 - 2x^2 + x$ .

## 11 Just a proof.

Give a *combinatorial proof* that

$$k \binom{n}{k} = n \binom{n-1}{k-1}.$$

If you give a non-combinatorial proof, half credit will be given.

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On the left-hand side, we can think of choose a  $k$ -set out of  $[n]$  and then choosing one of the *ranks* from 1 to  $k$ . If we remove the element of the chosen set with that rank, then what we are left with is a  $(k-1)$ -subset of a set of size  $n-1$ , since it know it won't contain that removed element, and we know that the removed element will be between 1 and  $n$ .

## 12 Just another proof.

Give a *combinatorial proof* that

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}.$$

*Hint:* It does not involve the binomial theorem. Remember that  $\binom{n}{k} = \binom{n}{n-k}$ .

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Following the hint, we have

$$\sum_{k=0}^n \binom{n}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k}.$$

So if we have  $2n$  things and we want to choose  $n$  of them (which represents the right-hand side of the initial equality), then it's the same as choosing some  $k$  of them,  $k$  between 0 and  $n$ , from the first  $n$  of the  $2n$ , and then we would choose  $n - k$  from the second  $n$  of the  $2n$  that precisely describes the left-hand sum.

