30 July 2009

Final Exam

Read these instructions carefully.

You have one hour, fifty minutes for the exam. Each of the questions are worth 10 points total.

You have your choice of doing either problem 1 or problem 2. You must do one of those two problems. In the table below, cross out the problem that you do **not** want graded.

Additionally, you have your choice of doing either problem 7 or problem 8. You must do one of those two problems. In the table below, cross out the problem that you do **not** want graded.

Finally, you have your choice of doing either problem 12 or problem 13. You must do one of those two problems. In the table below, cross out the problem that you do **not** want graded.

Problem	**Score**
1 2	
3	
4	
5	
6	
7 8	
9	
10	
11	
12 13	
Total:	

Cheat Sheet

The Twelvefold Way

balls (N)	urns (K)	Any $f : N \to K$	Injective $f : N \to K$	Surjective $f : N \to K$
distinguishable	distinguishable	k^n	$P(k, n)$	$k!S(n, k)$
indistinguishable	distinguishable	$\binom{k+n-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{k-1}$
distinguishable	indistinguishable	$S(n, 1) + S(n, 2) + \cdots + S(n, k)$	1, if $k \geq n$ 0, otherwise	$S(n, k)$
indistinguishable	indistinguishable	$p_1(n) + p_2(n) + \cdots + p_k(n)$	1, if $k \geq n$ 0, otherwise	$p_k(n)$

Power Series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$
$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots$$
$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \cdots$$
$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k = 1 + \binom{a}{1} x + \binom{a}{2} x^2 + \binom{a}{3} x^3 + \cdots$$

Inclusion-Exclusion

$$N(a_1' a_2' \cdots a_r') = S_0 - S_1 + S_2 - \ldots + (-1)^r S_r$$
$$e_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \ldots + (-1)^r \binom{m+(r-m)}{(r-m)} S_r$$
$$S_k = \sum N(a_{i_1} a_{i_2} \cdots a_{i_t})$$

where the last sum is taken over all choices of k distinct properties $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$.

Birkhoff's Formula

$$P(G, x) = \sum_{e,c} (-1)^e h(e, c) x^c$$

where the sum ranges over all possibilities for e, the number of edges, and c, the number of connected components.

1 Pick your poison: Sequences 1

The generating function for the sequence (a_k) is

$$F(x) = \frac{x+2}{(2x-1)(3x-1)}.$$

Find a closed-form formula for a_k.

Using partial fractions, we see that

$$\begin{aligned}
\frac{x+2}{(2x-1)(3x-1)} &= \frac{5}{2x-1} - \frac{7}{3x-1} \\
&= \frac{7}{1-3x} - \frac{5}{1-2x} \\
&= 7\left(\frac{1}{1-3x}\right) - 5\left(\frac{1}{1-2x}\right)
\end{aligned}$$

and so the formula will be

$$a_k = 7(3)^k - 5(2)^k.$$

2 Pick your poison: Sequences 2

A sequence (a_k) satisfies the recurrence

$$a_{k+2} = 5a_{k+1} - 6a_k$$

with initial conditions $a_0 = 2$, $a_1 = 11$. Give a closed-form expression for the generating function of (a_k).

Using power series, the recurrence transforms to

$$\sum_{k=0}^{\infty} a_{k+2}x^k = \sum_{k=0}^{\infty} a_{k+1}x^k - \sum_{k=0}^{\infty} 6a_kx^k.$$

Letting

$$F(x) = \sum_{k=0}^{\infty} a_kx^k,$$

it can be shown that

$$\begin{aligned} \sum_{k=0}^{\infty} a_{k+2}x^k &= \frac{F(x) - a_0 - a_1x}{x^2} \\ {}^{\infty}_{k=0} a_{k+1}x^k &= \frac{F(x) - a_0}{x} \end{aligned}$$

So we have

$$\begin{aligned} \frac{F(x) - a_0 - a_1x}{x^2} &= 5\left(\frac{F(x) - a_0}{x}\right) - 6F(x) \\ F(x) - 2 - 11x &= 5xF(x) - 10x - 6x^2F(x) \\ 6x^2F(x) - 5xF(x) + F(x) &= x + 2 \\ F(x) &= \frac{x + 2}{6x^2 - 5x + 1} \end{aligned}$$

3 Zing dolla dolla.

I open a savings account in the beginning of the year 0 with a 10% interest rate, compounded annually. I initially deposit \$1000 and at the beginning of each subsequent year I deposit another \$1000. Let a_k denote the amount of money I have in the bank account in the beginning of year k, *right after* I make the \$1000 deposit.

(a) Give a recurrence relation for (a_k).

(b) Find a closed-form formula for a_k.

Note: Interest compounded annually means that at the end of each year, interest is calculated (at the rate given) and then immediately added to the balance of the bank account.

We start with $a_0 = 1000$ and note that

$$a_k = 1.1a_{k-1} + 1000$$

since we get interest on the balance the year before, and then we deposit \$1000. We can write out a few terms, longhand:

$$\begin{aligned}
a_1 &= 1.1a_0 + 1000 \\
a_2 &= 1.1(1.1a_0 + 1000) + 1000 = (1.1)^2 a_0 + 1.1(1000) + 1000 \\
a_3 &= 1.1(1.1(1.1a_0 + 1000) + 1000) = (1.1)^3 a_0 + (1.1)^2(1000) + 1.1(1000) + 1000
\end{aligned}$$

and we see that, generally,

$$a_k = (1.1)^k a_0 + 1000(1 + 1.1 + (1.1)^2 + \cdots + (1.1)^{k-1}).$$

Through various means (generating functions, etc.) one can determine that

$$1 + 1.1 + (1.1)^2 + \cdots + (1.1)^{k-1} = 10((1.1)^k - 1)$$

and so

$$a_k = 1000(1.1)^k + 10((1.1)^k - 1).$$

4 The true life of a mathematician.

Every Sunday, Professor Raff goes to GAP to purchase shirts, one per day, for the next seven days. He always chooses between polo shirts, short-sleeved button-down shirts, and long-sleeved button-down shirts. He wants to wear at least one of each type of shirt.

(a) How many different ways are there to choose seven shirts for the upcoming week, assuming that different shirts of each type are indistinguishable?

(b) How many different ways are there to *assign* shirts to each day of the week, again assuming that different shirts of each type are indistinguishable?

(a) The number we desire is the coefficient of x^7 in

$$(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7)^3.$$

(b) The number we desire is the coefficient of $\frac{x^7}{7!}$ in

$$\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}\right)^3$$

Officially, each of our components in the generating function really only needed to go up to x^5, but it does not affect the answer.

5 Numb3rs.

How many numbers between 1 and 1000, inclusive, are divisible by neither 7 nor 23?

It may help to know that $\frac{1000}{7} \approx 142.857$ and $\frac{1000}{23} \approx 43.4783$.

Let

$$\begin{aligned} a_1 &= \text{property that we're divisible by 7} \\ a_2 &= \text{property that we're divisible by 23} \end{aligned}$$

Then

$$\begin{aligned} N(a_1'a_2') &= N - N(a_1) - N(a_2) + N(a_1a_2) \\ &= 1000 - 142 - 43 + 6 \\ &= 821 \end{aligned}$$

You were expected to know $(7)(23) = 161$ and $\lfloor \frac{1000}{161} \rfloor = 6$.

6 The magical life of trees.

Prove that if T is a tree, then T contains at least two vertices of degree one.

Hint: There are many ways to prove this. I have two suggestions. First suggestion: your proof can start with the sentence "Consider the longest path in T." Second suggestion: do a pigeonhole principle-style proof.

Officially, we need to make sure that our tree T has at least two vertices.

Consider the longest path in T. If an endpoint of this path had degree greater than 1, then it is adjacent to the next vertex on the path defined but it is also adjacent to a new vertex - if it was adjacent to another vertex on the path then that would create a cycle. Hence this contradicts the fact that we have the longest path. Hence, those two endpoints have degree 1.

7 Counterexample 1

Give an example of a graph that has an Eulerian cycle but does *not* have a Hamiltonian cycle.

Remember that Eulerian cycle is a misnomer; what I really mean to say is Eulerian closed chain

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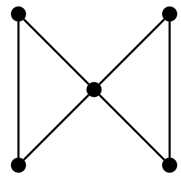
Here is one of many examples:

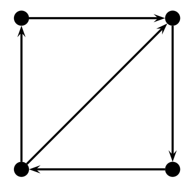


8 Counterexample 2

Recall that a directed graph is *strongly connected* if, given any two vertices v, w, there is a path from v to w and a path from w to v. The analogue for an Eulerian cycle (closed chain) for directed graphs is an *Eulerian closed path*.

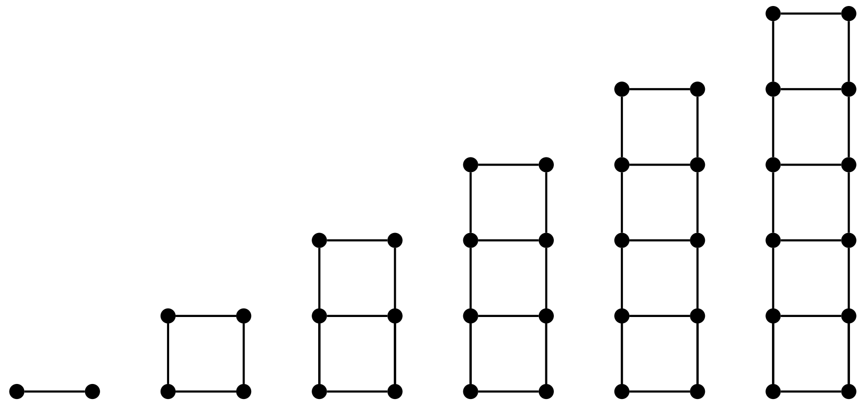
Give an example of a directed graph that is strongly connected but has no Eulerian closed path.

Again, here is but one of many examples:



9 Ladder Graphs

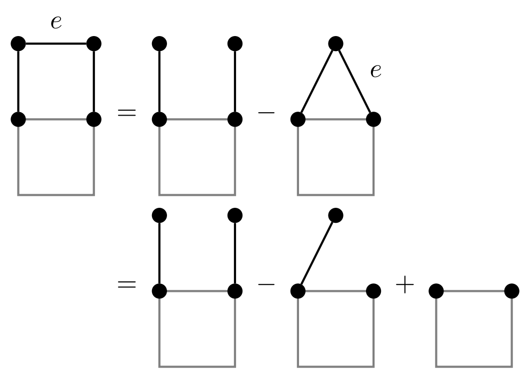
A *ladder graph* L_n has $2n$ vertices and hopefully the following examples for the first six ladder graphs will explain the definition of L_n.



L_1 L_2 L_3 L_4 L_5 L_6

Prove by induction that $P(L_n, x) = x(x-1)(x^2-3x+3)^{n-1}$.

Using the reduction formula, starting with L_{n+1}:



Hence, we conclude that

$$\begin{aligned}P(L_{n+1}, x) &= P(L_n, x)(x-1)^2 - P(L_n, x)(x-1) + P(L_n, x)\\ &= P(L_n, x)[(x-1)^2 - (x-1) + 1]\\ &= x(x-1)(x^2-3x+3)^{n-1}(x^2-3x+3)\\ &= x(x-1)(x^2-3x+3)^n\end{aligned}$$

and the proof is complete.

10 A quick break, part 1.

Our class (typically) has twelve students: eight men and four women. How many ways can a team of four students be made so that there is at least one woman on the team?

There are $\binom{12}{4}$ ways to make a group of 4. Of those, $\binom{8}{4}$ of them consist only of men. Hence, there are

$$\binom{12}{4} - \binom{8}{4}$$

ways to make a group of four with at least one woman.

11 A quick break, part 2.

A *dead man's hand*[1] in poker is a collection of five cards consisting of a pair of eights, a pair of aces, and an arbitrary fifth card. What is the probability that five cards dealt will be a dead man's hand?

Notes:

- A deck of cards consists of 52 cards, with four cards of each number from two to ten, plus jacks, queens, kings, and aces.
- Order does not matter in a poker hand.

We have to choose two of the four eights, two of the four aces, and 1 of the other 48 cards. Hence, the probability is

$$\frac{\binom{4}{2}\binom{4}{2}\binom{48}{1}}{\binom{52}{5}}.$$

[1]Named as such because it was the hand dealt to Wild Bill Hickok right before his murder.

12 Of course I am going to ask a combinatorial proof (part 1).

A *composition* or *unordered partition* of n is an *ordered* way to write n as the sum of non-zero positive integers. Give a combinatorial proof that the number of compositions of n is 2^{n-1}.

Hint: Think about counting the number of compositions with a fixed number of parts, and think about what kind of occupancy problem this is.

Consider n dots lined up in a row:

Notice that placing dividers in the $n-1$ spaces in between will allow us to read off a composition. Hence, we obtain a composition by choosing a subset of dividers. Conversely, given a subset of dividers we can read off a composition. Since there are 2^{n-1} subsets of the $n-1$ dividers, then there are 2^{n-1} compositions.

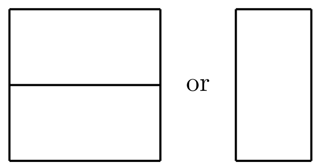
13 Of course I am going to ask a combinatorial proof (part 2).

Show that F_{n+1}, the Fibonacci number defined by

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2},$$

is the number of ways to tile a $2 \times n$ checkerboard with dominoes of size 2×1.

Consider a tiling of a $2 \times n$ checkerboard with these dominoes. The right end can be in two different configurations:



In the first configuration, what remains after removing those two dominoes is a tiling of a $2 \times (n-2)$ checkerboard. In the second configuration, what remains after removing that domino is a tiling of a $2 \times (n-1)$ checkerboard. Hence, the recurrence is satisfied, including initial conditions, so we're done.