13 July 2009

Homework 4 Problems - Final Version

It would do you well to use *Wolfram Alpha* for some of these problems, by typing, for example,
`expand (1+x+x^2)(1+x+x^3)`, or
`series of sin(x)cos(x)`.

5.1, #2f: Use the known Maclaurin expansions on page 286 to find the Maclaurin expansion of

$$\frac{1}{(1-x)^3}$$

by adding, composing, differentiating, and so on.

5.1, #5: Suppose that the ordinary generating function for the sequence (a_k) is given as follows. In each case, find a_3.

(a) $(x-7)^3$

(b) $\frac{14}{1-x}$

(c) $\ln(1-2x)$

(d) e^{5x}

5.2, #11: Make use of derivatives to find the ordinary generating function for the following sequences (b_k).

(a) $b_k = k^2$

(b) $b_k = k(k+1)$

(c) $b_k = (k+1)\frac{1}{k!}$

5.3, #7: Suppose there are p kinds of objects, each in infinite supply. Let a_k be the number of distinguishable ways of choosing k objects if and only if an even number (including 0) of each kind of object can be taken. Set up a generating function for a_k.

5.3, #17: Find the number of integer solutions to the following equations/inequalities:

(a) $b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 15$, with $0 \le b_i \le 3$ for all i.

(b) $b_1 + b_2 + b_3 = 15$, with, $0 \le b_i$ for all i and b_1 odd, b_2 even, and b_3 prime.

(c) $b_1 + b_2 + b_3 + b_4 = 20$, with $2 \le b_1 \le 4$ and $4 \le b_i \le 7$ for $2 \le i \le 4$.

(d) $b_1 + b_2 + b_3 + b_4 \leq 10$, with $0 \leq b_i$ for all i. (*Hint:* Include a "slack" variable b_5 to create an inequality.)

5.4, #9: Five different banks offer certificates of deposit (CDs) that can only be purchased in multiples of $1,000. If an investor has $10,000, in how many different ways can she invest in CDs?

5.4, #21: Three people each roll a die once. In how many ways can the score add up to 9?

5.5, #5: Find the number of 3-link RNA chains if the available bases are 2 A's, 3 G's, 3 C's, and 1 U. Check your answer by enumeration.

Extra problem. Recall that an *integer partition* of n is the way of writing n as a sum of positive integers. The number of different ways to do so is denoted by $p(n)$. We also have two more related definitions:

- $p_o(n)$ is the number of integer partitions of n so that each term in the sum is odd.
- $p_d(n)$ is the number of integer partitions of n so that each term in the sum is distinct.

For example, here is the list of all integer partitions of 7:

$$\begin{aligned}
7 &= 7 \\
7 &= 6 + 1 \\
7 &= 5 + 2 \\
7 &= 5 + 1 + 1 \\
7 &= 4 + 3 \\
7 &= 4 + 2 + 1 \\
7 &= 4 + 1 + 1 + 1 \\
7 &= 3 + 3 + 1 \\
7 &= 3 + 2 + 2 \\
7 &= 3 + 2 + 1 + 1 \\
7 &= 3 + 1 + 1 + 1 + 1 \\
7 &= 2 + 2 + 2 + 1 \\
7 &= 2 + 2 + 1 + 1 + 1 \\
7 &= 2 + 1 + 1 + 1 + 1 + 1 \\
7 &= 1 = 1 + 1 + 1 + 1 + 1 + 1
\end{aligned}$$

As you can see, $p_o(7) = p_d(7) (= 5)$. This is not a fluke! We will use generating functions to show that this is the case.

(a) Show that the generating function for $p_o(n)$ is

$$(1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^5+x^{10}+\cdots)\cdots.$$

(b) Show that the generating function for $p_d(n)$ is

$$(1+x)(1+x^2)(1+x^3)\cdots.$$

(c) Taking advantage of the fact that $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$ and that $(1+x^i) = \frac{1-x^{2i}}{1-x^i}$, show that the two generating functions from (a) and (b) are, in fact, equal. From this, explain why $p_e(n) = p_o(n)$.