20 July 2009

Homework 5 Problems - Final Version

6.1, # 16: Suppose that we have 10-cent stamps, 18-cent stamps, and 28-cent stamps, each in unlimited supply. Let $f(n)$ be the number of ways of obtaining n cents of postage if the order in which we put on stamps counts. For example, $f(10) = 1$ and $f(20) = 1$ (two 10-cent stamps), while $f(28) = 3$ (one 28-cent stamp, or a 10-cent stamp followed by an 18-cent stamp, or an 18-cent stamp followed by a 10-cent stamp).

(a) If $n > 29$, derive a recurrence for $f(n)$.

(b) *Use the recurrence* of part (a) to find the number of ways of obtaining 66 cents of postage.

(c) Check your answer to part (b) by writing down all the ways.

6.2, # 11: Consider the recurrence $b_k = 7b_{k-1} - 10b_{k-2}$. Which of the following sequences is a solution?

(a) 2^k

(b) $5^k - 2^k$

(c) $2^k + 7$

(d) $2^k - 5^k$

(e) $2^k + 5^{k+1}$

6.2, # 12: Use the method of characteristic roots to solve the following recurrences:

(a) $a_n = -2a_{n-1} - a_{n-2}$ with $a_0 = 2$, $a_1 = 2$.

(b) $b_k = -7b_{k-1} + 18b_{k-2}$ with $b_0 = 0$, $b_1 = 8$.

(d) $f_{n+1} = 2f_n + 3f_{n-1}$ with $f_0 = f_1 = 2$.

(e) $h_n = 9h_{n-2}$ with $h_0 = 4$, $h_1 = 2$.

6.3, # 2b: Use generating functions to solve the following recurrences under the given initial conditions:

$$a_{k+1} = 3a_k + 2, a_1 = 1.$$

7.1, # 9: A total of 100 students at a college were interviewed. Of these, 38 were taking a French course; 45 were taking a physics course; 28 a mathematics course; 25 a history course; 22 were taking French and physics; 23 French and mathematics; 10 physics and mathematics; 1 French and history; 21 physics and history; 14 mathematics and history; 11 French, physics, and mathematics; 8 French, physics, and history; 6 French, mathematics, and history; 6 physics, mathematics, and history; and 5 were taking courses in all four subjects. How many students were taking at least one course in the subjects in question?

7.2, # 12:

(a) If four fair coins are tossed, use Theorem 7.4 to compute the probability that there will be exactly 2 heads.

(b) Check your answer by computing it directly.

Extra Problem: The following problem has many different names, but we will call it the *best prize problem.*

The setup: There are n boxes, each with a different prize in it. Each prize has a different value, and each prize can be compared to all the other ones value-wise. You are allowed to open one box at a time, and when you open the box and see the prize, you can either take the prize and end the game (leaving all the unopened boxes unopened), or you can discard the prize and move to the next box. Once you discard a prize you are not allowed to get it back. You want a strategy for obtaining *the* best prize out of all n boxes.

Your strategy will be as follows: you will look in the first k boxes (the "test group") and discard all the prizes in them. You will then start opening the other $n-k$ boxes and you will keep the first prize you see that is better than the best one you saw in the test group (unless you get all the way to the end, in which case you choose the last prize).

First, some quick shots:

(a) *Using counting arguments*, show that in a list of $n-1$ distinct integers, the probability that the largest element in the list is in one of the first k positions is $\frac{k}{n-1}$.

(b) Show that there are $(n-1)!$ permutations of $[n]$ where the largest element n is at some predefined position i.

Fix a number k which will be used for our strategy. We will define W to be the set of permutations that let us win the largest prize when we follow the strategy of throwing away the first k boxes.

(c) Find W when $n = 4$ and $k = 2$.

Let B_i represent the set of permutations that have n in the i^{th} position. Notice that

$$W = \bigcup_{i=1}^{n} (W \cap B_i).$$

(d) Explain why $W \cap B_i = \emptyset$ for $1 \leq i \leq k$.

(e) Using (a) and (b), show that

$$|W \cap B_i| = \frac{k}{i-1}(n-1)!$$

and conclude that

$$|W| = \sum_{i=k+1}^{n} \frac{k}{i-1}(n-1)!$$

(f) Using the approximation

$$\sum_{i=1}^{n} \frac{1}{i} \approx \ln n,$$

show that

$$|W| \approx n! \left(\frac{k}{n} \ln \left(\frac{n}{k} \right) \right).$$

(g) Show that the quantity

$$\frac{k}{n} \ln \left(\frac{n}{k} \right)$$

is maximized when $\frac{k}{n} = \frac{1}{e}$.