25 June 2009

Quiz 1

---

1. How many odd numbers between 1000 and 9999 have distinct digits?

Since our number has to be odd, we should first figure out how many possibilities there are for the last digit. Then we have free rein over choosing the other four digits.

There are 5 ways to choose the last digit (either 1, 3, 5, 7, or 9). That digit can not be reused. Then there are 8 choices for the first digit (it can't be zero!), 8 choices for the second digit (since now we have the choice of zero) and 7 choices for the third digit. So the total possible number of ways is

$$(8)(8)(7)(5) = 2240.$$

2. Calculate the probability that if a DNA chain of length 5 is chosen at random, it will have at least four A's.

First off, there are 4^5 different ways to get a DNA chain of length 5. We will now count the number of chains with at least four A's, which we will get by counting the number of chains with exactly 4 A's and the number of chains with exactly 5 A's. For having exactly four A's, there are $\binom{5}{4} = 5$ ways to place the four A's, and then there are three choices for the other spot, for 15 ways total. Additionally, there is only one way you can have exactly 5 A's in the 5-length sequence. So there are 16 ways total to have at least four A's, so the probability is

$$\frac{16}{4^5}.$$

3. Consider the sum

$$\sum_{i=0}^{n} \binom{n}{i} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

Find out what the sum equals for $n = 2, 3, 4$, guess what the sum is equal to, and prove your answer by giving a combinatorial proof.

$$n = 2 : \binom{2}{0} + \binom{2}{1} + \binom{2}{2} = 1 + 2 + 1 = 4$$

$$n = 3 : \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8$$

$$n = 4 : \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 1 + 4 + 6 + 4 + 1 = 16$$

So we think the sum is equal to 2^n, and that is indeed the casy, for 2^n represents the number of subsets of $[n]$, and we get the sum by conditioning on the size of the subset. So a subset of $[n]$ has size exactly 0 OR size exactly 1 OR . . . OR size exactly n, and there is no overlap, so

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$