

Quiz 2

The Twelfold Way

balls (N)	urns (K)	Any $f : N \rightarrow K$	Injective $f : N \rightarrow K$	Surjective $f : N \rightarrow K$
distinguishable	distinguishable	k^n	$P(k, n)$	$k!S(n, k)$
indistinguishable	distinguishable	$\binom{k+n-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{k-1}$
distinguishable	indistinguishable	$S(n, 1) + S(n, 2) + \cdots + S(n, k)$	1, if $k \geq n$ 0, otherwise	$S(n, k)$
indistinguishable	indistinguishable	$p_1(n) + p_2(n) + \cdots + p_k(n)$	1, if $k \geq n$ 0, otherwise	$p_k(n)$

When it is unclear whether balls or cells are distinguishable, you should state your interpretation, give a reason for it, and then proceed.

1. An elevator with 11 passengers stops at 6 different floors. If we are interested only in the passengers who get off together, how many possible distributions are there?

Each passenger is a unique person, hence distinguishable. These are the balls. As stated, we only care about which which passengers get off together, so the urns, which are the floors, are indistinguishable. There are six urns, one for each floor, and we assume that at least one person gets off at each floor (it doesn't necessarily have to be the case, but we will make the assumption nevertheless). Hence we have $S(11, 6)$ ways of doing so.

2. Find the coefficient of x^9 in the expansion of $(2x + 4y)^{11}$.

Using the binomial theorem, we have

$$(2x + 4y)^{11} = \sum_{n=0}^{11} \binom{11}{n} (2x)^n (4y)^{11-n},$$

and the coefficient of x^9 occurs when $n = 9$, so we have

$$\binom{11}{9} (2x)^9 (4y)^2 = \binom{11}{9} 2^9 4^2 x^9 y^2$$

so the coefficient is $\binom{11}{9} 2^9 4^2$.

3. Suppose that there are 10 people at a party whose (integer) ages range from 0 to 100. Show that there are two distinct, but not necessarily disjoint, subsets of people that have exactly the same age. *Hint:* You will need to figure out how many different groups of people there are, and you will also need to find out what the maximum total age can be of a group.

With n people at a party, each person can either know $0, 1, 2, \dots$, or $n - 1$ people. This represents the pigeonholes, and there are n pigeonholes. This is a problem as it stands, for we could potentially have one person in each pigeonhole. However, that can't be the case, as there can't be a person who knows no one *and* a person who knows everyone at this party! Hence either the 0 pigeonhole or the $n - 1$ pigeonhole is empty, leaving n people and $n - 1$ pigeonholes.

Alternatively, you can first see if there is exactly one loner in the party who doesn't know anyone. If so, remove him. If there are no loners, then there is no 0 pigeonhole. If there is more than one loner, then we are done with the problem.