23 July 2009

Quiz 4

Power series:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$$

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k} x^k = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

1. If a_n represents the number of permutations of the set $[n]$, what is the *exponential* generating function for (a_n)?

There are $n!$ permutations of the set $[n]$, so the exponential generating function is

$$\sum_{n=0}^{\infty} n! \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

2. Verify that $(n2^n + 3 \cdot 5^n)$ is a solution to the recurrence

$$b_n = 9b_{n-1} - 24b_{n-2} + 20b_{n-3}.$$

The characteristic polynomial of this recurrence is

$$x^3 - 9x^2 + 24x - 20 = (x-5)(x-2)^2.$$

Hence, any linear combination of $2^n, n2^n$, and 5^n are solutions, of which $n2^n + 3 \cdot 5^n$ is one.

3. The generating function for some sequence (a_k) is

$$F(x) = -\frac{22 - 16x}{(1-2x)(5-3x)}.$$

Find a closed-form expression for a_k.

We use partial fractions:

$$\begin{aligned} -\frac{22-16x}{(1-2x)(5-3x)} &= \frac{A}{1-2x} + \frac{B}{5-3x} \\ 16x - 22 &= A(5-3x) + B(1-2x) \end{aligned}$$

When $x = \frac{5}{3}$ then

$$\begin{aligned} 16\left(\frac{5}{3}\right) - 22 &= B\left(1 - 2\left(\frac{5}{3}\right)\right) \\ \frac{14}{3} &= -\frac{7}{3}B \\ B &= -2 \end{aligned}$$

When $x = \frac{1}{2}$ then

$$\begin{aligned} -14 &= \frac{7}{2}A \\ A &= -4 \end{aligned}$$

So

$$\begin{aligned} -\frac{22-16x}{(1-2x)(5-3x)} &= \frac{-4}{1-2x} + \frac{-2}{5-3x} \\ &= -4\left(\frac{1}{1-2x}\right) - \frac{2}{5}\left(\frac{1}{1-\frac{3}{5}x}\right) \\ &= -4\left(\sum_{k=0}^{\infty}(2x)^k\right) - \frac{2}{5}\left(\sum_{k=0}^{\infty}\left(\frac{3}{5}x\right)^k\right) \end{aligned}$$

and so

$$a_k = -4(2)^k - \frac{2}{5}\left(\frac{3}{5}\right)^k.$$